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PRELIMINARY NUMERICAL RESULTS FOR CONSTRAINED NONLINEAR MULTIBODY SYSTEMS THROUGH AN ADAPTIVE TIME-DISCONTINUOUS GALERKIN SCHEME

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Abstract. *This paper is concerned with the time integration of the equations of motion of nonlinear multibody systems discretized using the finite element method. A time-discontinuous Galerkin scheme is used to obtain the response of the system. The Lagrange multiplier technique is used to enforce the kinematic constraints among the bodies. The formulation uses Cartesian coordinates to represent the position of each body with respect to an inertial system. The scheme is cast in a predictor-multicorrector form, which provides unconditional stability, third order accuracy and high frequency numerical dissipation. An adaptive time-stepping algorithm is used to select the proper time step size to accurately solve the system of equations. Numerical examples are presented to demonstrate the performance of the algorithm.*

Keywords: Nonlinear multibody systems, Structural Dynamics, Time-discontinuous Galerkin.

1. INTRODUCTION

In structural dynamic analyses of nonlinear constrained multibodies, the problem of solving for the transient response of the systems deals with dynamic equilibrium equations written in the form

$$M\ddot{u}(x,t) + N[u(x,t), \dot{u}(x,t)] \;=\; P[u(x,t), t]\,, \tag{1}$$

associated with kinematic constraints

$$C[u(x,t)] \;=\; 0\,, \tag{2}$$

with initial conditions

$$u(x,t_0) = u_0 \quad \text{and} \quad \dot{u}(x,t_0)] = v_0\,, \tag{3}$$

where M is the consistent finite element integrated mass matrix, N is the nonlinear internally resisting forces in the structure, which may depend on displacements and velocities, and P is the externally applied forces that vary in time, generally, but which may also depend on the displacements. The constraints that connect each body to one another are represented in Eq. (2) by C, a set of holonomic, or more specifically, scleronomic or stationary, equality constraints.

High frequencies are present in the system due to the geometric, displacements and rotations, nonlinearities. Moreover, the kinematic constraints introduce infinite frequencies into the system, which are of a purely numerical origin, i.e, there is no physical meaning related to them. Thus, one has to be extremelly cautious on selecting a time integration scheme to solve for the response of the system. The chosen scheme must be able of handling not only the high, but also the infinite, frequencies present in the numerical response. The widely accepted Newmark scheme (Newmark, 1959), despite its good characteristics of accuracy and unconditional stability, cannot be used onto the solution of the class of problems represented by Eqs. (1), (2), and (3). The reader if referred to Cardona (Cardona & Geradin, 1989) for details. A time integration scheme that can effectively be used on the solution of the equations of motion of multibody systems is the time-discontinuous Galerkin. Using the natural framework of the second-order hyperbolic equations, instead of relying on converting the equations to a first-order symmetric hyperbolic form, which is actually not always possible, Hulbert (1989) developed time and space-time discontinuous Galerkin finite element methods to solve structural dynamics and elastodynamics problems. The resulting systems of equations are larger than the original ones, and fully coupled, which increases the computational costs of the solutions. Nevertheless, the scheme may be cast in a predictor-multicorrector form, which alleviates the high computational costs of solving the fully coupled systems, maintains the high order of accuracy and unconditional stability of the original scheme, and improves its characteristics of high frequency numerical dissipation.

The work herein presented deals with an adaptive time-stepping procedure based on a time-discontinuous Galerkin scheme to solve for the transient response of nonlinear constrained multibody systems. A single-field formulation is used with displacements approximated as a quadratic function. Numerical examples are presented and the features of the algorithm are discussed.

2. A TIME-DISCONTINUOUS GALERKIN FINITE ELEMENT

This work concerns the analysis of nonlinear constrained multibody systems discretized using the finite element method. The formulation uses Cartesian coordinates to represent the position of each body with respect to an inertial frame. The Lagrange multiplier technique is used to enforce the kinematic constraints among the various bodies. This approach allows a modular development of finite elements to represent a variety of kinematic constraints, so that general multibody configurations can effectively be modeled. The resulting systems of equations are differential-algebraic in nature. Such systems are stiff due to the presence of high frequencies in the elastic members, and also the infinite frequencies associated with the kinematic constraints. In reality, no mass is associated with the Lagrange multipliers degrees of freedom resulting algebraic equations coupled to the differential equations of the system.

Consider (Hulbert, 1989) a partition of the time domain $I = (0, T)$ in the form $0 = t_0 < t_1 < \cdots < t_N = T$ with corresponding time steps $\Delta t_n = t_n - t_{n-1}$ and intervals

$I_n = (t_{n-1}, t_n)$. The finite element interpolation functions for the trial displacements are

$$S^h = \{\underline{u}^h \in \bigcup_{n=1}^{N} (P^2(I_n))^{n_{eq}}\} \tag{4}$$

where P^2 stands for second-order polynomial and each member of S^h is a vector consisting of n_{eq} quadratic functions on each time interval I_n. By construction, the interpolation functions are continuous within each time interval and may be discontinuous across time slabs. To enforce the continuity across time intervals a temporal jump operator

$$[\underline{w}(t_n)] = \underline{w}(t_n^+) - \underline{w}(t_n^-) \tag{5}$$

where

$$\underline{w}(t_n^{\pm}) = \lim_{\epsilon \to 0^{\pm}} \underline{w}(t_n + \epsilon) \tag{6}$$

is used. The displacement weighting function space is identical to the trial displacement's.

The statement (Hulbert, 1989) of the time-discontinuous Galerkin finite element method for the single-field formulation, applied to the ordinary differential equations associated with the semidiscrete form of linear elastodynamics is:

Find $\underline{u}^h \in S^h$ such that for all $\underline{w}^h \in W^h$

$$\begin{aligned} \int_{t_{n-1}^+}^{t_n^-} [\underline{\dot{w}}^h \cdot (M\underline{\ddot{u}}^h + C\underline{\dot{u}}^h + K\underline{u}^h - \underline{F})]dt \quad &+ \\ \underline{\dot{w}}^h(t_{n-1}^+) \cdot M[\underline{\dot{u}}^h(t_{n-1}^+) - \underline{\dot{u}}^h(t_{n-1}^-)] \quad &+ \\ \underline{w}^h(t_{n-1}^+) \cdot K[\underline{u}^h(t_{n-1}^+) - \underline{u}^h(t_{n-1}^-)] \quad &= \quad 0. \end{aligned} \tag{7}$$

In Eq. (7), $n = 1, 2, \cdots, N$, where N is the number of time intervals. Variables $\underline{u}^h$ and $\underline{w}^h$ are, respectively, displacements and weighting functions, $\underline{\dot{u}}^h$ and $\underline{\ddot{u}}^h$ are, respectively, velocities and accelerations. The last two terms on the left-hand side weakly enforce the initicial conditions for each time interval. These jump terms are stabilizing operators that have the effect of up-winding information with respect to time (Hulbert, 1989). Also, M, C and K are the mass, damping and stiffness matrices, respectively, and F is the force vector. Since the displacements are interpolated as quadratic functions, the resulting system of equations is three times larger than the ones solved by commonly used semidiscrete methods. To circumvent the high computational cost of solving the fully coupled equations, the system is cast in a predictor-multicorrector form, which maintains the characteristics of high order accuracy and unconditional stability of the scheme, besides improving its high frequency numerical dissipation capability.

3. THE ADAPTIVE TIME-STEPPING PROCEDURE

The response of constrained multibody systems often rapidly varies in time, indicating the need for an automated time step size adaptation procedure. Moreover, in modern structural dynamic analysis, in general, it is convenient that a time integration scheme allows automatic time step size control.

An adaptive time-stepping procedure, based on a time-discontinuous Galerkin scheme, for selecting the proper time step size is presented by Li & Wiberg (1996). They use a two-field formulation, namely the P1-P1 formulation (Hulbert, 1989), which interpolates displacements and velocities as piecewise linear funtions. In the study herein developed a

single-field formulation is used with displacements approximated as a quadratic function. The resulting systems of equations are smaller than the ones resulting from a two-field formulation with the advantage of improved accuracy characteristics (Hulbert, 1989).

The time adaptive algorithm presented by Li & Weiberg (1996) is applied to constrained nonlinear multibody systems. For the automatic time step size control the relative error at a time t_n is defined as

$$\epsilon_n = \left| \frac{E(t_n) - E(\overline{t})}{E(\overline{t})} \right| \tag{8}$$

where E is the total energy, $E(\cdot) = K(\cdot) + V(\cdot)$, i.e, the sum of kinetic and potential energies, and $E(\overline{t})$ is a reference energy of the system. It is expected that the relative errors satisfy the condition

$$\epsilon_n \leq \epsilon^{tol} \quad , \tag{9}$$

where ϵ^{tol} is a specified error tolerance. If requirement in Eq. (9) is not satisfied, a time step refinement is performed. The corresponding solution is rejected and given that the convergence rate for the algorithm is $O(\Delta t^3)$, the new time step size that will satisfy the error tolerance criterion is calculated, (Li & Wiberg, 1996) and (Wiberg & Li, 1994), as

$$\Delta t_n^{tol} = \left(\frac{\theta_t \, \epsilon^{tol}}{\epsilon_n} \right)^{1/3} \Delta t_n \quad , \tag{10}$$

where $\theta_t \leq 1.0$ is a reducing factor used to avoid the new predicted time step size being rejected. On the contrary, if the calculated error is much smaller than the tolerance ϵ^{tol}, i.e,

$$\epsilon_n < \gamma \, \epsilon^{tol} \tag{11}$$

the solution is accepted but the time step size may be increased according to Eq. (10) when the criterion in Eq. (11) is satisfied for a certain sucessive number of time steps. In Eq. (11) γ is a number much smaller than 1.0.

4. NUMERICAL EXAMPLES

4.1 Simple Pendulum

Consider a simple pendulum problem modeled as a point mass m and a massless rigid link of length ℓ. Two degrees of freedom u_x and u_y, respectively, vertical and horizontal displacements, describe the position of the mass m. Only gravity acts upon the system. Through an augmented Lagrange multiplier technique a constraint equation, namely $C = u_x^2 + u_y^2 - \ell^2 = 0$, that guarantees the constant length ℓ of the pendulum is added to the system. Such a constraint introduces an infinite frequency into the system of equations, which then becomes prone to numerical instabilities and oscillations of a purely numerical origin. Cardona & Geradin (1989) have shown the impossiblity of solving this problem with the Newmark time integration scheme. Nevertheless, the time-discontinuous Galerkin scheme efficiently solves this nonlinear constrained problem as shown by Damilano (1993).

Despite the reduced number of degrees of freedom and its rigid body nature, this problem is studied aiming the possibility of applying the technique herein described to more representative multibody systems, e.g., problems with large number of degrees of

freedom, several kinematic constraints and nonlinear elastic members. For the present study $m = 1.0$ kg, $\ell = 0.5$ m, $g = 9.81$ m/s^2, and the initial conditions are $u_x = 0.5$ m, $u_y = \dot{u}_x = 0.0$, $\dot{u}_y = -1.695$ m/s. The solution is calculated for 50 seconds and the numerical results are in excellent agreement with their analytical counterparts. Initially solved without adaptivity, i.e., with a constant Δt, a large increase in relative energy error is observed during the first 10 seconds of the solution, as shown in Figure (1). In fact, it increases about 3 orders of magnitude. This error keeps increasing during the computations of the solution, however, at a low gradient in the instants that follow the initial

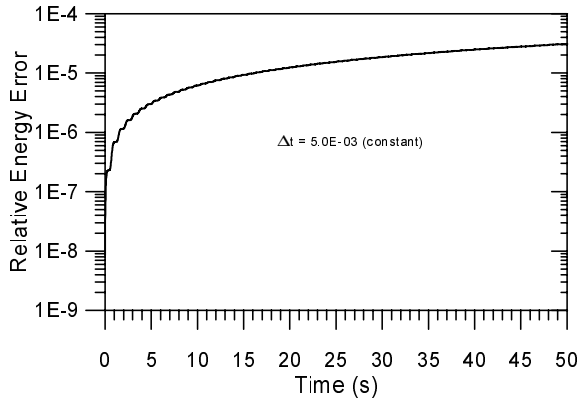

Figure 1: Relative energy error for the nolinear pendulum with constant time step.

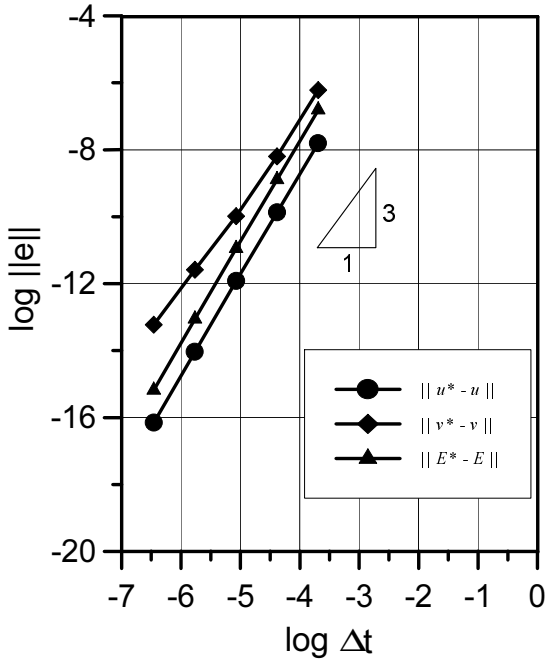

Figure 2: Rate of convergence at time $t = 10.0$ seconds for the nonlinear pendulum.

10 seconds. The inherent high frequency numerical dissipation of the time-discontinuous Galerkin method completely eliminates the undesired high frequency numerical oscillations. This numerical dissipation guarantees the stability of the scheme. However, it also implies the energy decaying results observed. A conventional analysis of the scheme based

on the characteristics of the amplification matrix (Hughes, 1992), for linear systems, results that the scheme is third-order accurate. However, there is no guarantee the same accuracy will be observed with nonlinear constrained systems. To assess the order of accuracy of the scheme applied to the nonlinear pendulum, the results at time $t = 10$ seconds were used to calculate the errors of the solution as functions of the time step size. The results are presented in Figure (2), where u, v, and E stand for displacements, velocities and total energy, respectively. In fact, the scheme is third-order accurate. However, there is no guarantee that the same high order of accuracy will be maintained by the scheme in the solution of different nonlinear problems. The time adaptive algorithm predicts the

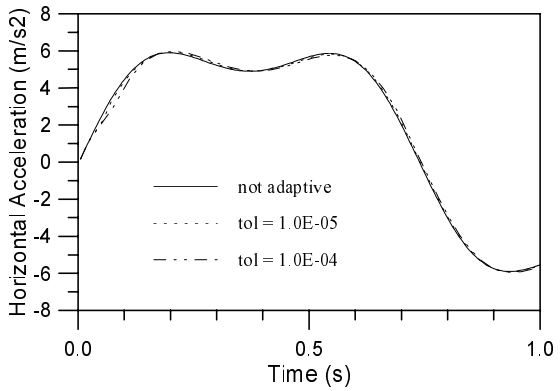

Figure 3: Horizontal accelerations for the simple pendulum.

solution for two different error tolerances, $1.0\text{x}10^{-05}$ and $1.0\text{x}10^{-04}$, with $\theta_t = 0.95$ and $\gamma = 0.6$ for both cases. Figure (3) shows the smooth results for horizontal accelerations at the first second of calculations. There are no high frequency numerical oscillations present in the response. The solution obtained without time adaptivity falls on top of its ana-

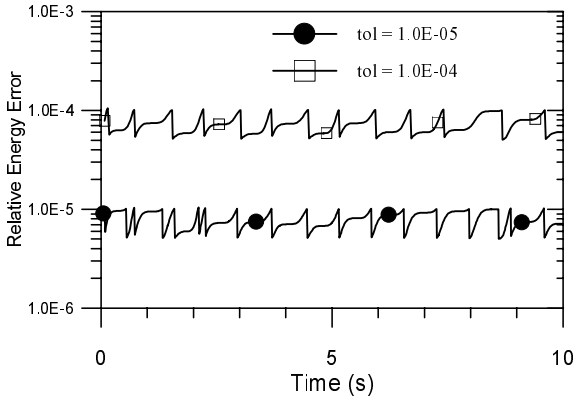

Figure 4: Error in energy for the simple pendulum.

lytical counterpart. The adaptive time-stepping routine allows time step changes right at the very first time step of integration, which explains the minor difference around $t = 0.1$ seconds. However, the scheme is capable of recovering from that slight error and brings the solution to coincide with the exact one. The estimated errors in energy for the initial 10 seconds of computation are presented in Fig. (4). It is important to remember that the numerical dissipation inherently present in the scheme results a total energy decay.

Evidently, it turns out to be impossible for the algorithm to use the same reference energy $E(\overline{t})$ in Eq. (8), throughout the entire computation of the solution. Thus, the calculation of the response starts using $E(\overline{t})$ equal to the initial total energy of the system. Then, if at a given time step the convergence criterion in Eq. (9) is not satisfied the time step size will be reduced. At each time step that Δt has to be reduced, this process goes on up to a maximum number of repetitions, or until Δt reaches a specified minimum value. In either case, if the error is not larger than $1.1\epsilon^{tol}$, the solution is accepted but the new energy of reference is the average between the reference energy at the previous time step and the energy computed at the present time step. Results for both analyses show the energy errors bounded by their respective limits. In both cases, the maximum value allowed for the time step size is $\Delta t = 5.0\text{x}10^{-02}$. A spectral analysis of the solution for constant Δt showed that $\Delta t = 3.7\text{x}10^{-02}$ is the largest time step size that could accurately integrate the equations of motion. Figure (5) shows that the algorithm keeps the time step size within the necessary limit for accuracy. During the entire period of the computations the

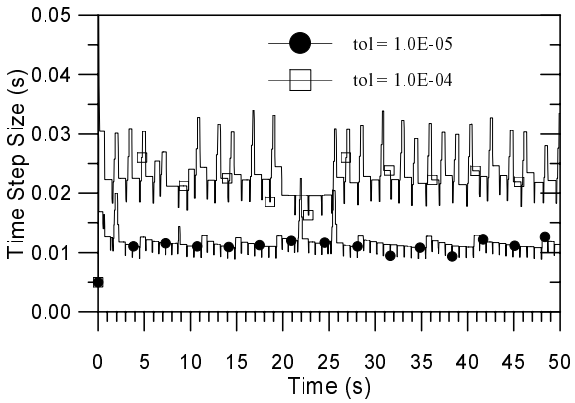

Figure 5: Time step size variation for the simple pendulum.

kinematic constraint, which guarantees the constant length of the pendulum, is satisfied at the order of machine accuracy, i.e., $1.0\text{x}10^{-15}$. The solution without adaptivity used 223 seconds of CPU time and 10000 time steps. The adptive algorithm with the error tolerance $1.0\text{x}10^{-05}$ used 57% of the computation time above and reduced the number of time steps to 4341. Relaxing the error tolerance to $1.0\text{x}10^{-04}$ results a reduction in time of computation to 36% of the computation time for a constant Δt, further reducing the number of time steps to 2195.

4.2 Slider Crank Mechanism

In order to assess the capability of the algorithm to overcome problems with singularities, a slider crank mechanism is analyzed. Figure (6) shows the mechanism, which for specific geometric configurations, and depending on the solution scheme to be used, rules out the numerical solution. For instance, if the method of coordinate partitioning is to be used, coordinates θ_1 and θ_2 are related as

$$\tan\theta_1 = \frac{1}{\tan\frac{\theta_2}{2}} \ . \tag{12}$$

Evidently, for specific positions of the masses, such as when $\theta_1 = 180$ deg. and $\theta_2 = -180$ deg., and $\theta_1 = 0$ deg. and $\theta_2 = 180$ deg., the solution will break down since Eq. (12)

cannot be solved for such configurations. The mechanism consists of two equal masses $m_1 = m_2 = 1.0$ kg, connected by means of two massless rods of equal length $\ell_1 = \ell_2 = \ell = 1.0$ m. Mass m_2 is constrained to frictionlessly move on a horizontal line such that

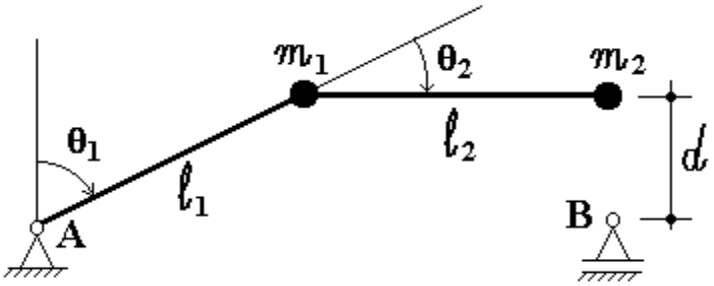

Figure 6: Slider crank mechanism.

the distance d remains equal to zero, which is the kinematic constraint imposed onto the system. The only force acting upon the system is gravity $g = 9.81$ m/s^2. The kinematic

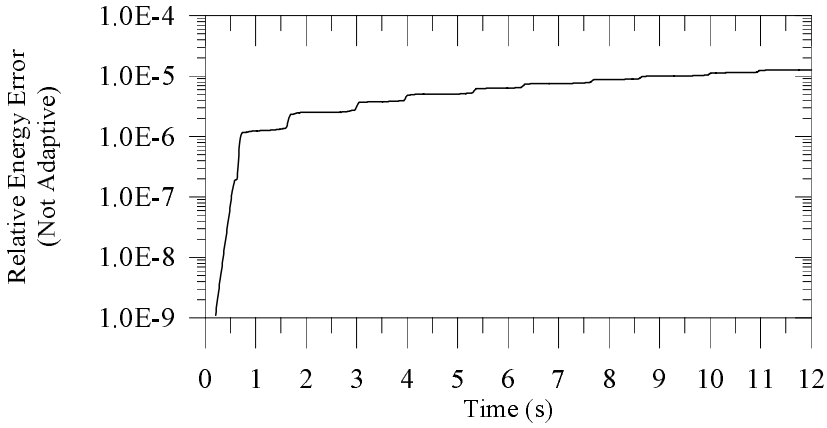

Figure 7: Relative energy error for the slider crank mechanism with constant time step.

constraint is expressed as $d = \ell[\cos\theta_1 + \cos(\theta_1 + \theta_2)]$. An augmented Lagrange multiplier technique is used to enforce the constraint upon the system. Initial conditions are $\theta_1 = 45$ deg., $\theta_2 = 90$ deg., and $\dot{\theta}_1 = \dot{\theta}_2 = 0$. The numerical results do not present any sort of high frequency vibration content, falling on top of their analytical counterparts. Once again, the relative energy error shows an energy decaying in the response, as pictured in Fig. (7). This energy decaying characteristic of the time-discontinuous Galerkin scheme results from its capability of dissipating any spurious high frequencies that may artificially be introduced into the system. As shown in the literature (Cardona & Geradin, 1989) accelerations of constrained systems are the responses most sensitive to spurious high frequencies oscillations and instabilities. Horizontal accelerations for masses m_1 and m_2 are depicted in Fig. (8). The results coincide with the analytical solutions and, as in the previous example, the infinite frequencies introduced into the system by the kinematic constraint, are completely dissipated. The error tolerance in energy for the analysis is $\epsilon^{tol} = 1.0\text{x}10^{-05}$. Since the limit tolerances are 1.1 ϵ^{tol} and 0.6 ϵ^{tol}, respectively for the maximum and minimum tolerances, Fig. (9) shows the relative energy error oscillating between the bounds defined by such limits. Finally, the variation in time step size along

the time integration process is presented in Fig. (10). The maximum size allowed for Δt is $2.0\text{x}10^{-01}$. It is clear, from Fig. (10), that the adaptive algorithm takes that value for Δt, after the first few steps of calculation. However, it rapidly comes back to a much smaller

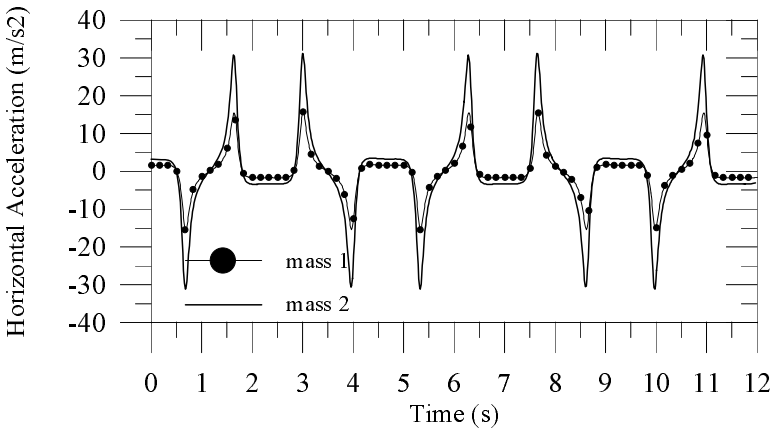

Figure 8: Horizontal accelerations for masses m_1 and m_2.

value and oscillates within the limits for accurately obtaining the response for the system. It can still be seen that the smallest values for Δt coincide, in time, with the regions where the accelerations curves present the steepest gradientes. Since accelerations and velocities,

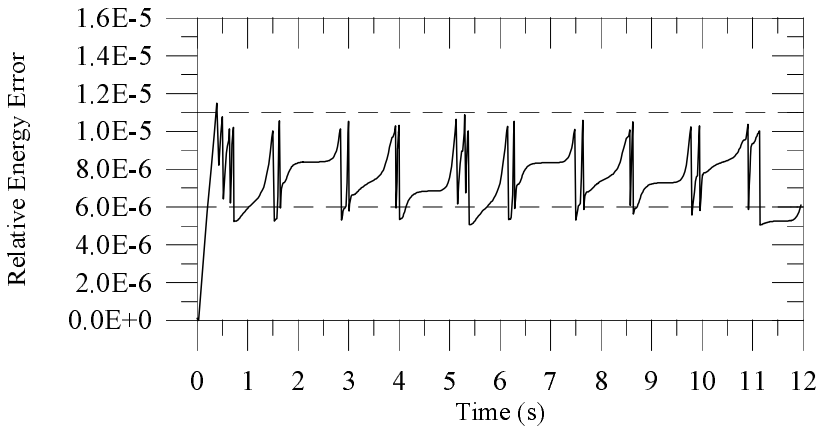

Figure 9: Relative energy error for the slider crank mechanism with $\epsilon^{tol} = 1.0\text{x}10^{-05}$.

along with displacements , either linear or rotational, play an important role in the time integration solution, the variation in time step size emphasizes the accuracy characteristics of the algorithm, i.e., it reduces the time step size where needed and enlarges it otherwise.

5. CONCLUSIONS

An adaptive time-stepping algorithm is applied to a time-discontinuous Galerkin scheme on the solution of nonlinear constrained multibody systems. The resulting systems of differential-algebraic equations are larger than the original ones, fully coupled, and prone to high frequency oscillations and instabilities of a purely numerical origin. The inherent high frequency numerical dissipation of the scheme completely eliminates the undesired instabilities. A single-field formulation of the time-discontinuous Galerkin

finite element scheme using quadratic interpolation functions produces coupled systems of equations that are three times larger than the original ones. The resulting systems are solved in a predictor-multicorrector form, which alleviates the high computational cost of solving the fully coupled systems, and improves the scheme's characteristics of accuracy and high frequencies numerical dissipation. The total energy of system is used as error

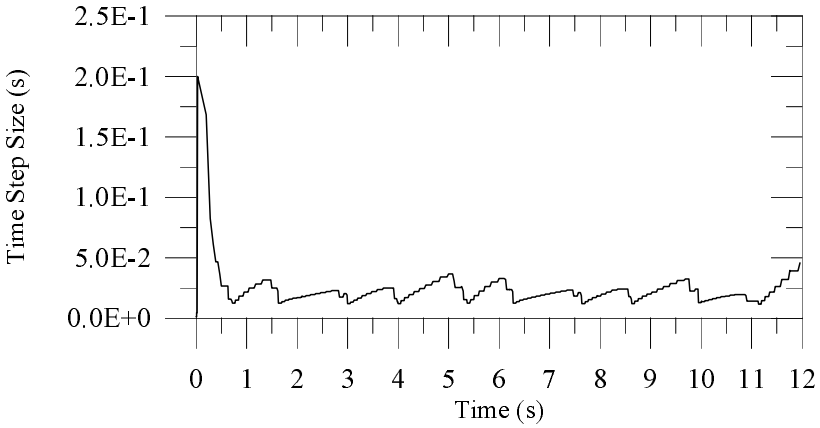

Figure 10: Temporal time step size for the slider crank mechanism.

measure and a time step adaptation routine is employed. The adaptive algorithm changes the time step size maintaining a priory specified accuracy for the solution. It was observed that in some analysis a significant reduction in computational cost was obtained. The scheme is not sensitive to configuration singularities that may occur during the motion of the structural system.

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